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Integrable hydrodynamics of Calogero–Sutherland model: bidirectional Benjamin–Ono equation

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Abstract
We develop a hydrodynamic description of the classical Calogero–Sutherland liquid: a Calogero–Sutherland model with an infinite number of particles and a non-vanishing density of particles. The hydrodynamic equations, being written for the density and velocity fields of the liquid, are shown to be a bidirectional analog of the Benjamin–Ono equation. The latter is known to describe internal waves of deep stratified fluids. We show that the bidirectional Benjamin–Ono equation appears as a real reduction of the modified KP hierarchy. We derive the chiral nonlinear equation which appears as a chiral reduction of the bidirectional equation. The conventional Benjamin–Ono equation is a degeneration of the chiral nonlinear equation at large density. We construct multi-phase solutions of the bidirectional Benjamin–Ono equations and of the chiral nonlinear equations.

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1. Introduction

The Calogero–Sutherland model (CSM) [1, 2] describes particles moving on a circle and interacting through an inverse sin-square potential. The Hamiltonian of the model reads

$$\mathcal{H}_{\mathrm{CSM}} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\left(\frac{\pi}{L}\right)^2 \sum_{j,k=1;\, j\neq k}^{N} \frac{g^2}{\sin^2 \frac{\pi}{L}(x_j - x_k)}, \tag{1}$$

where x_j are coordinates of N particles, p_j are their momenta and g is the coupling constant. We took the mass of the particles to be unity. The momenta p_j and coordinates x_j are canonically conjugate variables.

The model (classical and quantum) occupies an exceptional place in physics and mathematics, and has been studied extensively. It is completely integrable. Its solutions can be written down explicitly as finite dimensional determinants (for review see [3]).

In the limit of a large period $L \to \infty$ the CSM degenerates to its rational version—Calogero (aka Calogero–Moser) model (CM), where the pair–particle interaction is $1/x^2$.[4] The CSM itself is a degeneration of the elliptic Calogero model, where the pair–particle interaction is given by the Weierstrass $\wp$ function of the distance. In this paper, we discuss the classical trigonometric model (1) commenting on the rational limit when appropriate.

We are interested in describing a Calogero–Sutherland *liquid*, i.e., the system (1) in a thermodynamic limit when $N \to \infty$ and $L \to \infty$, while the average density N/L is kept constant. We assume that the limit exists and that in this limit a microscopic density and current fields

$$\rho(x,t) = \sum_{j=1}^{N} \delta(x - x_j(t)), \tag{2}$$

$$j(x,t) = \sum_{j=1}^{N} p_j(t)\delta(x - x_j(t)) \tag{3}$$

are smooth single-valued real periodic functions with a period L equal to the period of the potential[5]. In this case, the system will be described by hydrodynamic equations written on the density field $\rho(x,t)$ and the velocity field $v(x,t)$. The velocity is defined as $j = \rho v$.

The hydrodynamic approach is a powerful tool to study the evolution of smooth features with typical size much larger than the inter-particle distance. Apart from application to the CSM, the hydrodynamic equations obtained in this paper are interesting integrable equations. We show that they are new real reductions of the modified Kadomtzev–Petviashvili equation (MKP1).

In this paper we consider a classical system; however, the approach developed below can be extended to the quantum case $\{p_j, x_k\} = \delta_{jk} \to [p_j, x_k] = -\mathrm{i}\hbar\delta_{jk}$ almost without changes. For a brief description of the hydrodynamics of the quantum system see [4]. The hydrodynamics of the quantum Calogero model has been studied previously [5, 6] in the framework of the *collective field theory* and some of the results below can be obtained in a classical limit (see [7]) of the quantum counterparts of [5, 6].

The outline of this paper is as follows. In section 2, we parameterize the particles of the CSM as poles of auxiliary complex fields so that the motion of particles is encoded by evolution equations for fields. In section 3, we derive a hydrodynamic limit of these equations—continuity and Euler equations with a particular form of specific enthalpy. We will refer to these equations as to the bidirectional Benjamin–Ono equation or 2BO. We present the Hamiltonian form of 2BO in section 4. In section 5, we discuss the bilinear form of 2BO and its relation to MKP1. In section 6, we obtain the chiral nonlinear equation (CNL)—the chiral reduction of 2BO—and discuss some of its properties. In section 7, we construct multi-phase and multi-soliton solutions of 2BO and CNL as a real reduction of MKP1. These solutions correspond to collective excitations of the original many-body system. Some technical points are relegated to the appendices.

[4] In the rational case, one usually adds a harmonic potential, $\frac{1}{2}\omega^2\sum_i x_i^2$, to the Hamiltonian to prevent particles from escaping. This addition does not destroy the integrability of the system [2].

[5] It is likely that there are classes of solutions of the CSM, whose thermodynamic limit consists of a number of interacting liquids. In this case the microscopic density give rises to a number of functions in the continuum—the densities of the distinct interacting liquids. In this paper we consider a class of solutions which leads to a single liquid.

2. Particles as poles of meromorphic functions

The equations of motion of the CSM are readily obtained from the Hamiltonian (1):

$$\dot{x}_j = p_j, \tag{4}$$

$$\dot{p}_j = -g^2 \frac{\partial}{\partial x_j} \sum_{k=1(k\neq j)}^{N} \left(\frac{\pi}{L} \cot \frac{\pi}{L}(x_j - x_k) \right)^2. \tag{5}$$

We rewrite this system in an equivalent way as

$$\mathrm{i}\frac{\dot{w}_j}{w_j} = \frac{g}{2}\left(\frac{2\pi}{L}\right)^2 \left(\sum_{k=1}^{N} \frac{w_j + u_k}{w_j - u_k} - \sum_{k=1(k\neq j)}^{N} \frac{w_j + w_k}{w_j - w_k} \right), \qquad j = 1, \ldots, N, \tag{6}$$

$$-\mathrm{i}\frac{\dot{u}_j}{u_j} = \frac{g}{2}\left(\frac{2\pi}{L}\right)^2 \left(\sum_{k=1}^{N} \frac{u_j + w_k}{u_j - w_k} - \sum_{k=1(k\neq j)}^{N} \frac{u_j + u_k}{u_j - u_k} \right), \qquad j = 1, \ldots, N, \tag{7}$$

where $w_j(t) = \mathrm{e}^{\mathrm{i}\frac{2\pi}{L}x_j(t)}$ are complex coordinates lying on a unit circle, while $u_j(t) = \mathrm{e}^{\mathrm{i}\frac{2\pi}{L}y_j(t)}$ are auxiliary coordinates. Indeed, differentiating (6) with respect to time and using (6) and (7) to remove first derivatives in time one obtains equations equivalent to (4) and (5).

We note that while the coordinates x_j are real, i.e., $|w_j| = 1$, the auxiliary coordinates, $y_j(t)$, are necessarily complex. Given initial data as real positions and velocities $x_j(0)$ and $\dot{x}_j(0)$, one can find complex y_j from (6) and then initial complex velocities $\dot{y}_j(0)$ from (7). Once x_j and $\dot{x}_j$ are chosen to be real, they will stay real at later times, even though coordinates y_i are moving in a complex plane.

The coordinates $w_j(t)$ and $u_j(t)$ determine an evolution of two functions

$$u_1(w) = g\frac{\pi}{L}\sum_{j=1}^{N} \frac{w + w_j}{w - w_j} = -\mathrm{i}g \sum_{j=1}^{N} \frac{\pi}{L} \cot \frac{\pi}{L}(x - x_j), \qquad w = \mathrm{e}^{\mathrm{i}\frac{2\pi}{L}x}, \tag{8}$$

$$u_0(w) = -g\frac{\pi}{L}\sum_{j=1}^{N} \frac{w + u_j}{w - u_j} = \mathrm{i}g \sum_{j=1}^{N} \frac{\pi}{L} \cot \frac{\pi}{L}(x - y_j), \qquad w = \mathrm{e}^{\mathrm{i}\frac{2\pi}{L}x}. \tag{9}$$

The latter functions play a major role in our approach. These are rational functions of w regular at infinity and having particle coordinates as simple poles with equal residues $2\pi g/L$.

The condition that the coordinates of particles x_j are real yields Schwarz reflection condition for the function u_1 with respect to the unit circle

$$\overline{u_1(w)} = -u_1(1/\bar{w}) \qquad \text{or} \qquad \overline{u_1(x)} = -u_1(\bar{x}), \tag{10}$$

where the bar denotes complex conjugation. The values of $u_1(w)$ in the interior and exterior of a unit circle are related by Schwarz reflection.

Comparing (6), (4) and (9) we note that while the function $u_1(w)$ encodes the positions of particles w_j, the function $u_0(w)$ encodes the momenta of particles as its values at particle positions w_j

$$p_j = u_0(w_j) + g\frac{\pi}{L}\sum_{k=1(k\neq j)}^{N} \frac{w_j + w_k}{w_j - w_k}. \tag{11}$$

We note here that the positions of the particles fully determine the imaginary part of the field u_0 on a unit circle. Indeed, we have from (11)

$$\mathrm{Im}\, u_0(x_j) = g \sum_{k\neq j} \frac{\pi}{L} \cot \frac{\pi}{L}(x_j - x_k). \tag{12}$$

We now introduce complex functions

$$u = u_0 + u_1, \qquad \tilde{u} = u_0 - u_1. \tag{13}$$

One can show that they obey the equation

$$u_t + \partial_x \left[\frac{1}{2}u^2 + \mathrm{i}\frac{g}{2}\partial_x \tilde{u}\right] = 0. \tag{14}$$

Indeed, substituting the *pole ansatz* (8) and (9) into (14) and comparing the residues at poles w_j and u_j one arrives at (6) and (7).

Equation (14) connects two complex functions u_0 and u_1. The equation is equivalent to the *modified Kadomtzev–Petvisashvili* equation (or simply MKP1). We will discuss its relation to MKP1 in section 5.

However, being complemented by the Schwarz reflection condition (10), analyticity requirements and an additional reality requirement it becomes an equation uniquely determining u_0 and u_1 through their initial data.

The analyticity requirements read: $u_0(w)$ is analytic in a neighborhood of a unit circle $|w| = 1$, while u_1 is analytic inside $|w| < 1$ and outside $|w| > 1$ of the unit circle, approaching a constant at $w \to \infty$. An additional reality requirement is the relation between the imaginary part of u_0 on a unit circle and u_1 stemming from condition (12). We formulate and discuss these conditions in sections 3.3 and 5.

We will refer to equation (14) as the bidirectional Benjamin–Ono equation (2BO). It is a bidirectional (having both right- and left-moving waves) generalization of the conventional *Benjamin–Ono* equation (BO) arising in the hydrodynamics of stratified fluids [8]. We discuss its hydrodynamic form in the following section.

The solution of (14) given by (8) and (9) is the CSM many-body system with a finite number of particles (1). Other solutions describe CSM fluids. They are the central issue of this paper.

To conclude this section we make the following comment. The function u_1 can be expressed solely in terms of the microscopic density of particles (2) as

$$u_1(w) = -\pi g \oint \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \frac{\zeta + w}{\zeta - w} \rho(\zeta). \tag{15}$$

The integral in this formula goes over the unit circle $\zeta = \exp\left(\mathrm{i}\frac{2\pi}{L}x\right)$. In the following, we will denote for brevity $\rho(\zeta)$ as $\rho(x)$, when ζ lies on a unit circle $\zeta = \mathrm{e}^{\mathrm{i}\frac{2\pi}{L}x}$. The density itself can be obtained as a difference of limiting values of the field u_1 at the real x (on the unit circle). The discontinuity of u_1 on the unit circle gives a microscopic density (2) of particles

$$u_1(x + \mathrm{i}0) - u_1(x - \mathrm{i}0) = -2\pi g \rho(x), \qquad \mathrm{Im}\, x = 0, \quad 0 < \mathrm{Re}\, x < L. \tag{16}$$

3. Hydrodynamics of Calogero–Sutherland liquid

3.1. Density and velocity

We assume that in the thermodynamic limit $N, L \to \infty$, $N/L = \mathrm{const}$ the poles of the function u_1 are distributed along the real axis with a smooth density $\rho(x)$ and consider a complex field

$u_1(w)$ given by formula (15). Note that $u_1(w)$ defined by (15) is analytic everywhere outside of the real axis of x (everywhere off the unit circle in the z-plane) approaching a constant as $z \to \infty$. It also satisfies the reality condition (10) (the density $\rho(x)$ is real). In the thermodynamic limit, the function u_1 is not a rational function anymore. It is discontinuous across the real axis with the discontinuity related to the density of particles by (16). The value of the field $u_1(x)$ on a real axis (on a unit circle in the z-plane) depends on whether one approaches the real axis from above or below (unit circle from the interior $z \to e^{i\frac{2\pi}{L}(x+i0)}$ or from the exterior $z \to e^{i\frac{2\pi}{L}(x-i0)}$). More explicitly, we have from (15)

$$u_1(x \pm i0) = \pi g(\mp\rho + i\rho^H). \tag{17}$$

The superscript H in the second term of (37) denotes the Hilbert transform and is defined as (see appendix A for definitions and some properties of the Hilbert transform)

$$f^H(x) = \fint_0^L \frac{dy}{L} f(y) \cot \frac{\pi}{L}(y - x). \tag{18}$$

We also assume that in $N \to \infty$ limit, the complex field $u_0(w)$ remains analytic in the vicinity of the real axis in the x-plane (i.e., in the vicinity of a unit circle in the z-plane).

The 2BO (14) does not explicitly depend on the number of particles N. It holds also in the thermodynamic limit $N, L \to \infty$, $N/L = \text{const}$; however, solutions describing a liquid are no longer rational functions.

We can use 2BO to define velocity through the *continuity equation*

$$\rho_t + \partial_x(\rho v) = 0. \tag{19}$$

The discontinuity of the complex field $u(x)$ (13) across the real axis as well as a discontinuity of the field u_1 (see (16)) is the density

$$u(x + i0) - u(x - i0) = u_1(x + i0) - u_1(x - i0) = -2\pi g\rho(x). \tag{20}$$

Differentiating (20) with respect to time and using 2BO (14) we obtain the continuity equation and identify the *velocity field* $v(x)$ as

$$\begin{aligned} v(x) &= u_0(x) + \tfrac{1}{2}\,(u_1(x + i0) + u_1(x - i0)) - ig\partial_x \log \sqrt{\rho}(x) \\ &= u_0(x) + ig(\pi\rho^H(x) - \partial_x \log \sqrt{\rho}(x)) \end{aligned} \tag{21}$$

or

$$u_0(x) = v - ig(\pi\rho^H - \partial_x \log \sqrt{\rho}). \tag{22}$$

Since $v(x)$ is a real field, (22) provides a reality condition analogous to (12). Indeed, one can see from (22) that

$$\operatorname{Im} u_0(x) = -g(\pi\rho^H - \partial_x \log \sqrt{\rho}), \tag{23}$$

i.e., the imaginary part of $u_0(x)$ is completely determined by the density of particles or equivalently by the field u_1. It is also convenient to have an expression for $u(x)$ on a real axis

$$u(x \pm i0) = v + g(\mp\pi\rho + i\partial_x \log \sqrt{\rho}). \tag{24}$$

It has the same discontinuity across the real axis as $u_1(x)$.

3.2. Hydrodynamic form of 2BO

Now we are ready to cast equation (14) into a hydrodynamic form.

Taking the real part of 2BO (14) on the real axis and using identifications (17) and (22) and the continuity equation, (19), after some algebra we arrive at the *Euler equation*

$$v_t + \partial_x \left(\frac{v^2}{2} + w(\rho) \right) = 0, \tag{25}$$

with specific (per particle) enthalpy or chemical potential[6] given by

$$w(\rho) = \frac{1}{2}(\pi g \rho)^2 - \frac{g^2}{2} \frac{1}{\sqrt{\rho}} \partial_x^2 \sqrt{\rho} + \pi g^2 \rho_x^H. \tag{26}$$

Equations (19) and (25) are the continuity and Euler[7] equations of the classical Calogero–Sutherland model. They are the classical analogs of quantum hydrodynamic equations that have been obtained for the quantum CSM in [5, 6, 9] first using a collective field theory approach [10–12] and later by the *pole ansatz* similar to that used above [4]. It was noted in [12] and then in [7] that the system (19), (25) and (26) has a lot of similarities with the classical Benjamin–Ono equation [13]. The similarities and differences with the Benjamin–Ono equation are discussed below. We will refer to (19), (25) and (26) as to a hydrodynamic form of the *bidirectional Benjamin–Ono equation* (2BO).

3.3. Bidirectional Benjamin–Ono equation (2BO)

Let us now summarize the 2BO equation:

$$u_t + \partial_x \left[\frac{1}{2} u^2 + \mathrm{i} \frac{g}{2} \partial_x \tilde{u} \right] = 0, \tag{27}$$

$$u = u_0 + u_1, \qquad \tilde{u} = u_0 - u_1. \tag{28}$$

The functions u_0 and u_1 are subject to analyticity conditions

$$u_1(x) \quad \text{analytic for } \mathrm{Im}(x) \neq 0, \tag{29}$$

$$u_0(x) \quad \text{analytic for } |\mathrm{Im}(x)| < \epsilon \text{ for some } \epsilon > 0, \tag{30}$$

and to reality conditions

$$\overline{u_1(x)} = -u_1(\bar{x}). \tag{31}$$

In addition, the fact that equation (27) holds in the upper half-plane and in the lower half-plane (inside and outside of the unit circle) yields the condition

$$\mathrm{Im}[u(x \pm \mathrm{i}0)] = \frac{g}{2} \partial_x \log \mathrm{Re}[u_1(x \pm \mathrm{i}0)]. \tag{32}$$

It also follows from (17), (23) and (24). Condition (32) looks more 'natural' in the bilinear formulation (see, equation (55)).

These reality and analyticity conditions reduce two complex fields u_0 and u_1 to two real fields—density $\rho(x)$ and velocity $v(x)$ as (17) and (22). Then, a complex equation (14)

[6] The specific enthalpy and chemical potential are identical at zero temperature.

[7] Equation (25) has a form of an Euler equation for an isentropic flow. Because of the long-range character of interactions the enthalpy cannot be replaced by the conventional pressure term $\partial_x w(\rho) \to \rho^{-1} \partial_x (p(\rho))$—the standard form of the Euler equation.

defined in both half-planes immediately yields the hydrodynamic equations (19), (25) and (26). Inversely, knowing real periodic fields $\rho(x)$ and $v(x)$ one can find fields u_0, u_1 everywhere in a complex x-plane.

Mode expansion. The analyticity and reality conditions can be recast in the language of mode expansions. It follows from (15) that

$$u_1(w) = \begin{cases} -\pi g\left(\rho_0 + 2\sum_{n=1}^{\infty}\rho_n w^n\right), & |w| < 1, \\ \pi g\left(\rho_0 + 2\sum_{n=1}^{\infty}\rho_n^{\dagger} w^{-n}\right), & |w| > 1, \end{cases} \tag{33}$$

where $\rho_n = \rho_{-n}^{\dagger} = \int_0^L \frac{\mathrm{d}x}{L}\rho(x)\,\mathrm{e}^{-\mathrm{i}\frac{2\pi n}{L}x}$ are the Fourier components of the density.

The values of the field $u_1(w)$ in the upper and lower half-planes are then automatically related by Schwarz reflection (10).

Conversely, the field $u_0(x)$ being analytic in a strip around the unit circle is represented by Laurent series

$$u_0(w) = V_0 + \sum_{n=1}^{\infty}(a_n w^n + b_n w^{-n}), \qquad |\mathrm{Im}\log w| < 2\pi\epsilon/L. \tag{34}$$

The 2BO equation remains intact in the case of rational degeneration. Rational degeneration of formulae of section 2 is obtained by a direct expansion in $1/L$. In this limit, fields are defined microscopically as $u_1(x) = -\mathrm{i}g\sum_j \frac{1}{x-x_j}$ and $u_0(x) = \mathrm{i}g\sum_j \frac{1}{x-y_j}$.

4. Hamiltonian form of 2BO

The 2BO is a Hamiltonian equation. Let us start with its Hamiltonian formulation in the hydrodynamic form $\rho_t = \{H, \rho\}$, $v_t = \{H, v\}$ with the canonical Poisson bracket of density and velocity fields

$$\{\rho(x), v(y)\} = \delta'(x-y). \tag{35}$$

Equations (19), (25) and (26) follow from

$$H = \int \mathrm{d}x\left(\frac{\rho v^2}{2} + \rho\epsilon(\rho)\right), \tag{36}$$

$$\epsilon(\rho) = \frac{g^2}{2}(\pi\rho^H - \partial_x\log\sqrt{\rho})^2. \tag{37}$$

Here the 'internal energy' (37) and the enthalpy (26) are related by a general formula $w(\rho) = \frac{\delta}{\delta\rho(x)}\int \mathrm{d}x\,\rho\epsilon(\rho)$.

For references, we will give alternative expressions for the Hamiltonian. Let $\Psi = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}\vartheta}$ where $v = g\partial_x\vartheta$ then

$$H = \frac{g^2}{2}\int |\partial_x\Psi - \pi\rho^H\Psi|^2\,\mathrm{d}x, \tag{38}$$

where $\rho = |\Psi|^2$. The Poisson brackets for $\Psi(x)$ are canonical: $\{\Psi(x), \Psi(y)\} = 0$, and $\{\Psi(x), \Psi^{\star}(y)\} = \frac{\mathrm{i}}{g}\delta(x-y)$. The equations of motion for Ψ and $\Psi^{\star}$ are

$$\frac{\mathrm{i}}{g}\partial_t\Psi = \left[-\frac{1}{2}\partial_x^2 + \frac{\pi^2}{2}|\Psi|^4 + \pi(|\Psi|^2)_x^H\right]\Psi \tag{39}$$

and its complex conjugate. A simple change of a dependent variable $\Phi = \Psi\, e^{i\pi \int^x dx' |\Psi(x')|^2}$ leads to

$$\frac{i}{g}\partial_t \Phi = \left[-\frac{1}{2}\partial_x^2 + i2\pi (|\Phi|^2)_x^+\right]\Phi, \tag{40}$$

where f^+ denotes the function analytical in the upper half-plane of x defined as $f^+ = \frac{f - i f^H}{2}$. One can recognize in (40) the intermediate nonlinear Schrödinger equation (INLS) which appeared in [14] as an evolution of the modulated internal wave in a deep stratified fluid. Therefore, one can alternatively think of 2BO as the hydrodynamic form of (40) identifying hydrodynamic fields ρ and v to be

$$\Phi = \sqrt{\rho}\exp\left\{\frac{i}{g}\int^x dx'(v + \pi g \rho)\right\} \tag{41}$$

or with the field $u(x)$ from (24) as

$$i g \partial_x \log \Phi^\star = u(x - i0). \tag{42}$$

The Hamiltonian (36) or (38) can be rewritten in terms of Φ as

$$H = \frac{g^2}{2}\int |\partial_x \Phi - i2\pi \rho^+ \Phi|^2\, dx, \tag{43}$$

where $\rho = |\Phi|^2$. However, the Poisson brackets for Φ are no longer canonical[8].

2BO is an integrable system. It has infinitely many integrals of motion. The first three of them follow from global symmetries. They are conventional the number of particles $N = \int dx\, \rho$, the total momentum $P = \int dx\, \rho v$ and the total energy $H = \int dx \left(\frac{\rho v^2}{2} + \rho\epsilon(\rho)\right)$. They are conveniently written in terms of the fields u and $\tilde{u}$ as

$$I_1 = N = \frac{1}{2\pi g}\oint_C dx\, u, \tag{44}$$

$$I_2 = P = \frac{1}{2\pi g}\oint_C dx \frac{1}{2}u^2, \tag{45}$$

$$I_3 = 2H = \frac{1}{2\pi g}\oint_C dx \left[\frac{1}{3}u^3 + i\frac{g}{2}u\partial_x \tilde{u}\right], \tag{46}$$

where the integral is taken over both sides of the unit circle. ('double' contour C shown in figure 1). For more details on conserved integrals see appendix B.

The Poisson bracket for the fields $u_0(w)$ and $u_1(w)$ can be easily obtained from (17), (22) and (35) by analytic continuation. We find that $\{u_0(w), u_0(w')\} = \{u_1(w), u_1(w')\} = 0$ and

$$\{u_0(w), u_1(w')\} = ig\left(\frac{2\pi}{L}\right)^2 \frac{ww'}{(w - w')^2} = ig\partial_x \frac{\pi}{L}\cot\frac{\pi}{L}(x - y). \tag{47}$$

5. Bilinearization and relation to MKP1 equation

The equations described in the previous section, their integrable structures and their connection to integrable hierarchies are the most transparent in the bilinear form.

[8] Simple calculation using (35) gives $\{\Phi(x), \Phi(y)\} = \frac{\pi}{g}\Phi(x)\Phi(y)\,\mathrm{sgn}(x - y)$, $\{\Phi(x), \Phi^\star(y)\} = \frac{i}{g}\delta(x - y) - \frac{\pi}{g}\Phi(x)\Phi^\star(y)\,\mathrm{sgn}(x - y)$ and similar expressions for complex conjugated fields. One should think of $\Psi(x)$ as a canonical bosonic field while of $\Phi(x)$ as a classical analog of a field with fractional statistics.

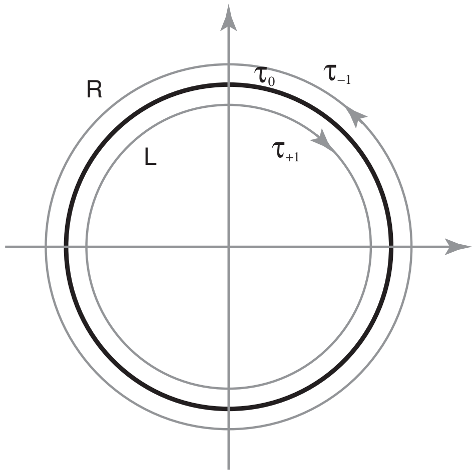

Figure 1. Contour C surrounding the unit circle is shown together with our conventions in defining right and left fields.

Let us introduce tau-functions τ_0 and τ_1 as

$$u_0 = \mathrm{i}g\partial_x \log \tau_0, \qquad u_1 = -\mathrm{i}g\partial_x \log \tau_1. \tag{48}$$

It can be easily checked that the 2BO (14) can be rewritten as an elegant bilinear Hirota equation on τ-functions:

$$\left(\mathrm{i}D_t + \frac{g}{2}D_x^2\right)\tau_1 \cdot \tau_0 = 0. \tag{49}$$

Here we used the Hirota derivative symbols defined as

$$D_x^n f(x) \cdot g(x) \equiv \lim_{y\to x}(\partial_x - \partial_y)^n f(x)g(y). \tag{50}$$

For example,

$$\begin{aligned} &D_t f \cdot g = (\partial_t f)g - f(\partial_t g), \\ &D_x^2 f \cdot g = \left(\partial_x^2 f\right)g - 2(\partial_x f)(\partial_x g) + f\left(\partial_x^2 g\right). \end{aligned} \tag{51}$$

We emphasize that the bilinear equation holds on both sides of the unit circle. Introducing notations

$$\tau_{\pm 1} = \tau_1(x \pm \mathrm{i}0) \tag{52}$$

we can rewrite the equation as

$$\begin{aligned} &\left(\mathrm{i}D_t + \frac{g}{2}D_x^2\right)\tau_{+1} \cdot \tau_0 = 0, \\ &\left(\mathrm{i}D_t + \frac{g}{2}D_x^2\right)\tau_{-1} \cdot \tau_0 = \left(-\mathrm{i}D_t + \frac{g}{2}D_x^2\right)\tau_0 \cdot \tau_{-1} = 0. \end{aligned} \tag{53}$$

Equation (49) is the modified Kadomtsev–Petviashvili equation (MKP1). MKP1 contains two independent functions τ_1 and τ_0 and is formally not closed. The analyticity and reality conditions (29)–(32), stemming from the fact that all solutions are determined by two real functions $\rho(x, t)$ and $v(x, t)$, close the equation. Under these conditions the equations can be seen as a real reduction of MKP1. Let us formulate these conditions in terms of tau-functions.

The first requirement is that $\tau_{\pm 1}$ is analytic and does not have zeros for $\mathrm{Im}\, x > 0$ (<0) after analytic continuation. Also τ_0 should be analytic and should not have zeros in the vicinity of the real axis, i.e., for $|\mathrm{Im}\, x| < \epsilon$ for some $\epsilon > 0$.

The second requirement is that $\tau_{\pm 1}$ should be related by Schwarz reflection (10). In terms of tau-functions it becomes on the unit circle (for real x)

$$\tau_{-1} = \overline{\tau_{+1}}\, \mathrm{e}^{\mathrm{i}\Theta(t)}, \tag{54}$$

where a phase $\Theta(t)$ can be any time-dependent function.

The third requirement is related to the fact that $\mathrm{Im}\, u_0$ is a function of density only and, therefore, can be expressed in terms of u_1 as can be easily seen from (17) and (22). This condition (32) can be written in a bilinear form as follows:

$$\mathrm{i}D_x \tau_{+1} \cdot \overline{\tau_{+1}} = A\tau_0\overline{\tau_0}. \tag{55}$$

The multiplicative constant A on the rhs of (55) fixes the relative normalization of τ_0 and τ_1 and is arbitrary. Condition (55) can be thought of as a part of Bäcklund transformation from the solution (τ_{+1}, τ_0) to the solution (τ_{-1}, τ_0) of MKP1 (53) [15].

Finally, we note that the pole ansatz solution (8) and (9) corresponds to the polynomial form of tau-functions with zeros at w_j and u_j:

$$\tau_1(w,t) = w^{-N/2}\prod_{j=1}^{N}(w - w_j(t)), \tag{56}$$

$$\tau_0(w,t) = w^{-N/2}\prod_{j=1}^{N}(w - u_j(t)). \tag{57}$$

6. Chiral fields and chiral reduction

6.1. *Chiral fields and currents*

The 2BO equation can be conveniently expressed through yet another right-handed and left-handed chiral field

$$J_{R,L} = v \pm g[\pi\rho + \partial_x(\log\sqrt{\rho})^H]. \tag{58}$$

These fields are real[9]. In terms of them, the 2BO equation (14) reads

$$\partial_t J_{R,L} + \partial_x\left(\frac{J_{R,L}^2}{2} \pm \frac{g}{2}\partial_x J_{R,L}^H\right) \mp g\partial_x[J_{R,L}\partial_x(\log\sqrt{\rho})^H - (J_{R,L}\partial_x \log\sqrt{\rho})^H] = 0. \tag{59}$$

Here ρ is a function of J_R and J_L implicitly given by (58). The Hamiltonian acquires a Sugawara-like form

$$H = \frac{1}{8}\int \mathrm{d}x\, \rho\left[(J_R + J_L)^2 + \left(J_R^H - J_L^H\right)^2\right] \tag{60}$$

with Poisson brackets

$$\{J_{R,L}(x), J_{R,L}(y)\} = \pm 2\pi g\partial_x\delta(x-y) \pm \frac{g}{2L}\partial_x\partial_y\left[\left(\frac{1}{\rho(x)} + \frac{1}{\rho(y)}\right)\cot\frac{\pi}{L}(x-y)\right], \tag{61}$$

[9] $J_{R,L}$ can be expressed solely in terms of u_0 field. It is easy to check that (58) is equivalent to $J_{R,L} = \mathrm{Re}\left(u_0 \mp \mathrm{i}u_0^H\right)$.

$$\{J_R(x), J_L(y)\} = -\frac{g}{2L}\partial_x\partial_y\left[\left(\frac{1}{\rho(x)} - \frac{1}{\rho(y)}\right)\cot\frac{\pi}{L}(x-y)\right]. \tag{62}$$

We note that Poisson brackets become canonical and left and right fields decouple in the limit of a constant density.

6.2. *Chiral reduction*

We first note that the right and left currents $J_{R,L}$ are not separated in equation (59). The equations for J_R and J_L are coupled through the density ρ which should be found in terms of $J_{R,L}$ from (58). However, it is possible to find the *chiral reductions* of 2BO assuming that one of the currents is constant. We explain this reduction in some detail in this section.

The 2BO (14) or (59) admits an additional reduction to a chiral sector [16] where one of the chiral currents (58), say left current, is a constant $J_L(x,t) = v_0 - \pi g\rho_0$. We can always choose a coordinate system moving with velocity v_0. This is equivalent to setting the zero mode of velocity to zero $v_0 = 0$. The condition $J_L = -\pi g\rho_0$ becomes

$$v = g[\pi(\rho-\rho_0) + \partial_x(\log\sqrt{\rho})^H]. \tag{63}$$

Then the currents can be expressed in terms of the density field only

$$J_L(x) = J_0, \qquad J_R(x) = J_0 + J(x), \tag{64}$$

$$J_0 = \pi g\rho_0, \qquad J(x) = 2g[\pi(\rho-\rho_0) + \partial_x(\log\sqrt{\rho})^H]. \tag{65}$$

It follows from equation (59) that once the current J_L is chosen to be constant $J_L(x) = J_0$ at $t = 0$, it remains constant at any later time. Condition (63), therefore, is compatible with 2BO. Then the density $\rho(x,t)$ evolves according to the continuity equation (19) with velocity determined by the density according to (63). We obtain an important equation (written in the coordinate system moving with velocity v_0)

$$\rho_t + g[\rho(\pi(\rho-\rho_0) + \partial_x(\log\sqrt{\rho})^H)]_x = 0. \tag{66}$$

We refer to this equation as the chiral nonlinear equation (CNL). A substitution of the chiral constraint (64) to (60) gives the Hamiltonian for CNL

$$H = \frac{1}{8}\int \mathrm{d}x\, \rho[J^2 + (J^H)^2] \tag{67}$$

with Poisson brackets for $J(x)$ following from (61). This equation constitutes one of the major results of this paper.

CNL can be written in several useful forms. One of them is

$$\varphi_t + g\left[\pi\rho_0\,(2e^{\varphi} - \varphi) + \frac{1}{2}\varphi_x^H\right]_x + \frac{g}{2}\varphi_x\varphi_x^H = 0, \tag{68}$$

where $\rho(x) = \rho_0\,\mathrm{e}^{\varphi(x)}$.

6.3. *Holomorphic chiral field*

Under the chiral condition (63) the field u_0 becomes analytic inside the disk. Indeed, combining (63) and (22) we obtain

$$u_0(w) = \frac{1}{2}\oint\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\frac{\zeta+w}{\zeta-w}J(\zeta), \qquad |w| < 1. \tag{69}$$

In the chiral case, it has only non-negative powers of w in the expansion (34). Negative modes vanish $b_n = 0$. Conversely, the condition of u_0 to be analytic inside the unit disk is equivalent to $J_L = \mathrm{const}$.

The current itself (64) is the boundary value of the field $\mathrm{Re}\, u_0$ harmonic inside the disk

$$J(x) = 2J_0 + 2\,\mathrm{Re}\, u_0 = 2J_0 + \sum_{n=1^\infty} (a_n w^n + \bar{a}_n w^{-n}). \tag{70}$$

The fields u and $\tilde{u}$ are in turn also analytic inside the disk. Let φ be a harmonic function inside the disk with the boundary value $\log(\rho/\rho_0)$. Then $\varphi = \phi(w) + \overline{\phi(w)}$, where $\phi(w) = (\log(\rho/\rho_0))^+$. Here $f^+(w) = \frac{1}{2}\int \frac{\zeta+w}{\zeta-w} f(\zeta)\frac{\mathrm{d}\zeta}{2\pi i\zeta}$ is a function analytic in the interior of a unit circle the value of which on the boundary of the disk is $(f(x) - \mathrm{i}f^H(x))/2$. It follows from (24) and (63) that

$$u = -J_0 + \mathrm{i}g\partial\phi, \qquad |w| < 1, \tag{71}$$

$$\tilde{u} = u + 4\pi g\rho_0\,(e^\varphi)^+\,, \qquad |w| < 1. \tag{72}$$

Then 2BO (27) becomes an equation on an analytic function in the interior of a unit circle

$$\dot{\phi} + \mathrm{i}\frac{g}{2}[(\partial\phi)^2 + \partial^2\phi] + \pi g\rho_0\partial\,(2e^\varphi - \varphi)^+ = 0. \tag{73}$$

This is the 'positive part' of (68) which is a direct consequence of (66).

We remark here that the chiral equation (66) has a geometric interpretation as an evolution equation describing the dynamics of a contour on a plane. Within this interpretation the term $\partial_x(\log\sqrt{\rho})^H$ of (66) is the curvature of the contour (see appendix C).

6.4. *Benjamin–Ono equation*

Another form of the chiral equation (66) arises when one considers the fields u and $\tilde{u}$ outside the disk. There neither u nor $\tilde{u}$ are analytic, but their boundary values are connected by the Hilbert transform

$$u(x - \mathrm{i}0) = -J_0 + 2g[\pi\rho + \mathrm{i}\partial_x(\log\sqrt{\rho})^+], \tag{74}$$

$$\tilde{u}(x - \mathrm{i}0) = -J_0 - \mathrm{i}u^H(x - \mathrm{i}0). \tag{75}$$

The bidirectional equation (27) complemented by this condition becomes unidirectional (chiral)

$$u_t + \partial_x\left[\frac{1}{2}u^2 + \frac{g}{2}\partial_x u^H\right] = 0. \tag{76}$$

This is just another form of the chiral equation (66).

The chiral equation (76) has the form of the Benjamin–Ono equation [13]. There are noticeable differences, however. Contrary to the Benjamin–Ono equation, equation (76) is written on a complex function, whose real and imaginary values at real x are related by conditions (74) implementing the reality of the density:

$$\mathrm{Re}\, u = -J_0 + 2g\pi\rho + g(\partial_x\log\sqrt{\rho})^H, \qquad \mathrm{Im}\, u = g\partial_x\log\sqrt{\rho}. \tag{77}$$

One understands this relation as a condition on the initial data. Once it is imposed by choosing the initial data for the density ρ, the condition remains intact during the evolution.

However, in the case when the deviation of a density is small with respect to the average density $|\rho - \rho_0| \ll \rho_0$, the imaginary part of u vanishes in the leading order of $1/\rho_0$ expansion

$$u \approx J_0 + 2\pi g\varphi \approx 2\pi g(\rho - \rho_0) + J_0,$$

and condition (77) becomes non-restrictive. In this limit, equation (76) becomes an equation on a single real function. It is the conventional Benjamin–Ono equation. One can think of CNL (66) as of finite amplitude extension of BO. Similarly, 2BO is an integrable bidirectional finite amplitude extension of BO. It is interesting that there exists another bidirectional finite amplitude extension of BO—the Choi–Camassa equation [17]. However, it seems that the latter is not integrable.

7. Multi-phase solution

In this section, we describe the most general finite dimensional solutions of 2BO. These are multi-phase solutions and their degenerations—multi-soliton solutions. In the former case, the τ-functions are polynomials of $\mathrm{e}^{\mathrm{i}k_i x}$, where k_i is a finite set of parameters, the latter are just polynomials of x. These solutions are given by determinants of finite dimensional matrices. They appeared in the arXiv version of [18]. One can construct those solutions using the transformation (41) of 2BO to INLS (40). For the latter multi-phase solutions were written in [14] (see also [19, 20]). We use a different route in this section deriving multi-phase and multi-soliton solutions as a real reduction of the corresponding solutions for MKP1.

7.1. Multi-phase and multi-soliton solutions of MKP1

We start from a general multi-phase solution of the MKP1 equation and then restrict it to the 2BO equation.

A general multi-phase solution of the MKP1 equation

$$\left(\mathrm{i}D_t + \frac{g}{2}D_x^2\right)\tau_1\cdot\tau_0 = 0 \tag{78}$$

is given by the following determinant formulae [22, 23],

$$\tau_a = \mathrm{e}^{\mathrm{i}\eta_a}\det\left[\delta_{jk} + c_{a,j}\frac{\mathrm{e}^{\mathrm{i}\theta_j}}{p_j - q_k}\right], \qquad a = 0, 1, \tag{79}$$

$$\frac{c_{1,j}}{c_{0,j}} = \frac{q_j}{p_j}, \tag{80}$$

where the phases are

$$g\theta_j(x,t) = (q_j - p_j)(x - x_{0j} - (K + v_0)t) - \frac{q_j^2 - p_j^2}{2}t, \tag{81}$$

$$g\eta_0(x,t) = Kx - \frac{K^2}{2}t - \left((v_0 + K)x - \frac{(v_0 + K)^2}{2}t\right), \tag{82}$$

$$g\eta_1(x,t) = Kx - \frac{K^2}{2}t. \tag{83}$$

This solution is characterized by an integer number N (the number of 'phases') and by $4N - 1$ parameters $p_j, q_j, c_{0,j}, x_{0j}$ and moduli K and v_0. The solutions become single valued on a unit circle if $p_j - q_j$ are integers in units of $g\frac{2\pi}{L}$.

7.2. Multi-phase solution of 2BO

Without further restrictions the parameters entering (79)–(83) are general complex numbers. Reality nature of the 2BO equation restricts them to be real.

The real moduli K and v_0 are obviously zero modes of the fields u_1 and u_0 respectively, and therefore, they are zero modes of the density $\rho_0 = \frac{1}{L}\int\rho\,\mathrm{d}x = -K/(\pi g)$ and velocity $\frac{1}{L}\int v\,\mathrm{d}x = v_0$.

7.2.1. Schwarz reflection condition. We have to restrict the coefficients $c_{a,j}$, so that there exists another solution τ_{-1}, τ_0 of equation (78) sharing the same τ_0 with the solution (85) and (86) and obeying the Schwarz reflection property (54).

The Galilean symmetry of equation (78) is here to help. If $\tau_a(x,t), a = 0, 1$ give a solution of (78) then the pair $\mathrm{e}^{\mathrm{i}P_a x - \mathrm{i}E_a t}\tau_a(x - gP_a t, t)$ is also a solution provided that $E_{a+1} - E_a = \frac{1}{2}(P_{a+1} - P_a)^2$.

Being applied to the solution (79)–(83) the Galilean invariance can be utilized as follows. We note from (81) that

$$\theta_j(x - Pt, t; \{p_j, q_j\}) = \theta_j(x, t; \{p_j + P, q_j + P\}). \tag{84}$$

Performing the Galilean boost to (79), multiplying both tau-functions by $\mathrm{e}^{-\mathrm{i}Pv_0 t}$ and shifting $p_j \to p_j - P, q_j \to q_j - P$ we obtain that

$$\tau_{-1} = \mathrm{e}^{\mathrm{i}\theta_1 - \mathrm{i}P(K+v_0)t + \frac{\mathrm{i}}{g}\left(Px - \frac{P^2}{2}t\right)} \det\left[\delta_{jk} + b_j \frac{\mathrm{e}^{\mathrm{i}\theta_j}}{p_j - q_k}\right], \tag{85}$$

$$\frac{b_j}{c_{0,j}} = \frac{q_j - P}{p_j - P} \tag{86}$$

form a solution of (78) with the same τ_0 (79).

Now we will show that for a particular choice of coefficients $c_{1,j}$ the Galilean boosted solution (85) and (86) is a complex conjugate of τ_{+1} from (79). To show this we will employ the determinant identity (D.7).[10]

We apply the determinant identity (D.7) to (85) and obtain

$$\tau_{-1} = \mathrm{e}^{\mathrm{i}\eta_1 - \frac{\mathrm{i}}{g}P(K+v_0)t + \frac{\mathrm{i}}{g}\left(Px - \frac{P^2}{2}t\right)} \left(\prod_j \mathrm{e}^{\mathrm{i}\theta_j}\sqrt{\frac{b_j}{\tilde{b}_j}}\right) \det\left[\delta_{jk} + \tilde{b}_j \frac{\mathrm{e}^{-\mathrm{i}\theta_j}}{p_j - q_k}\right], \tag{87}$$

where

$$\tilde{b}_j b_j = (p_j - q_j)^2 \prod_{k\neq j} \frac{(p_j - q_k)(q_j - p_k)}{(p_j - p_k)(q_j - q_k)}. \tag{88}$$

The Schwarz reflection condition (54) $\tau_{-1} = \overline{\tau_{+1}}\,\mathrm{e}^{\mathrm{i}\Theta(t)}$ requires

$$\tilde{b}_j = c_{1,j}, \tag{89}$$

gives a relation

$$P = -2K + \sum_j (p_j - q_j) \tag{90}$$

and determines $\Theta(t)$ as

$$\Theta(t) = \frac{1}{2g}t\left[-P^2 + 2K^2 + 4v_0 K + \sum_{j=1}^{N}\left(p_j^2 - q_j^2\right)\right]. \tag{91}$$

Finally, combining all relations (80), (86), (88) and (89) together we obtain the condition on coefficients $c_{a,j}$

$$\left(\frac{c_{a,j}}{p_j - q_j}\right)^2 \prod_{k\neq j} \frac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)} = \left(\frac{p_j}{q_j}\right)^{1-2a} \frac{p_j - P}{q_j - P}. \tag{92}$$

[10] A similar trick was used by Matsuno [19] to prove the reality of a multi-phase solution for the conventional Benjamin–Ono equation.

Condition (92) is necessary to turn a general solution of MKP1 into a solution of 2BO. We can check that the determinant formulae (79) with (92) satisfy the reality condition (55) with normalization constant

$$A = P\prod_{j=1}^{N}\frac{q_j}{p_j}. \tag{93}$$

We also have to find a condition that guarantees that τ_1 has no zeros inside the unit disk. Before turning to this analysis, we first discuss degeneration of formulae (79) into a multi-soliton solution.

7.2.2. Multi-soliton solution of 2BO. The multi-soliton solution of 2BO follows from the multi-phase solution in the limit $p_j \to q_j$. We introduce

$$k_j = p_j - q_j, \tag{94}$$

$$v_j = \tfrac{1}{2}(p_j + q_j) + K + v_0 \tag{95}$$

and consider the limit $k_j \to 0$ keeping v_j fixed. After some straightforward calculations we obtain

$$\tau_a = \mathrm{e}^{\mathrm{i}\eta_a}\det\left[\delta_{jk}(x - x_{0j} - v_j t + \mathrm{i}A_{a,j}) + \mathrm{i}g\frac{1-\delta_{jk}}{v_j - v_k}\right], \tag{96}$$

$$A_{a,j} = \frac{g}{2}\left(\frac{1}{v_j - v_0 + K} \pm \frac{1}{v_j - v_0 - K}\right), \qquad a = 0, 1. \tag{97}$$

One notes that in the limit $t \to +\infty$ the solution (96) asymptotically goes to the factorized form

$$\tau_a \to \mathrm{e}^{\mathrm{i}\theta_a}\prod_j (x - x_{0j} - v_j t + \mathrm{i}A_{a,j}), \tag{98}$$

describing separated single solitons.

Equation (98) gives a large time value of zeros of τ_1. Their imaginary part is

$$-\mathrm{Re}\, A_{1,j} = g\frac{K}{(v_j - v_0)^2 - K^2}. \tag{99}$$

It must be negative in order for τ_1 to have no zeros inside the unit disk. Since $K < 0$ we must require

$$(v_j - v_0)^2 > K^2. \tag{100}$$

Using $K = -\pi g\rho_0$ we rewrite (100) as a condition on soliton velocities

$$v_j < v_0 - \pi g\rho_0 \qquad \text{or} \qquad v_j > v_0 + \pi g\rho_0. \tag{101}$$

We stress here that the multi-soliton solution (96) is not chiral. Some of the soliton velocities can be negative (left inequality in (101)) while the rest might have positive velocities according to the right inequality in (101). In the following paragraph, we argue that under condition (101) and additional restrictions on parameters p_j, q_j (see equation (113) below) the moving zeros never cross the real axis, and therefore zeros stay outside of the unit disk at all times[11].

[11] In quantum problem the condition similar to (101) has a very straightforward interpretation: momenta of quasiparticles lie outside of the filled Fermi sea. Solitons in classical problem represent quasiparticles of the quantum model while quantum quasiholes become sound waves at the classical level [30].

To conclude this section we note a unique property of the 2BO equation (shared with the BO equation). Namely, there is a 'quantization' of the mass of solitons: each soliton of 2BO carries a unit of mass regardless of its velocity. We have for *N*-soliton solution

$$\int \mathrm{d}x(\rho - \rho_0) = N. \tag{102}$$

The total momentum and the total energy of a multi-soliton solution are given by

$$\int \mathrm{d}x(\rho v - \rho_0 v_0) = \sum_j v_j, \tag{103}$$

$$\int \mathrm{d}x \left(\frac{\rho v^2}{2} + \rho\epsilon(\rho) - \frac{\rho_0 v_0^2}{2} - \rho_0\epsilon(\rho_0) \right) = \sum_j \frac{v_j^2}{2}, \tag{104}$$

where $\epsilon(\rho)$ is defined in (37).

One-soliton solution has the form

$$\rho = \rho_0 + \frac{1}{\pi} \frac{A_1}{\xi^2 + A_1^2}, \qquad v = v_0 + g \frac{A_0}{\xi^2 + A_0^2}, \tag{105}$$

where

$$\xi = x - x_{01} - v_1 t \tag{106}$$

and

$$\begin{aligned} A_1 &= \mathrm{Re}\, A_{1,1} = \frac{\pi g^2 \rho_0}{(v_1 - v_0)^2 - (\pi g \rho_0)^2}, \\ A_0 &= \mathrm{Re}\, A_{0,1} = \frac{g(v_1 - v_0)}{(v_1 - v_0)^2 - (\pi g \rho_0)^2}. \end{aligned} \tag{107}$$

This one-soliton solution has been found first in [7] (see also [24]). The one-soliton solution (105) and (107) is written in terms of *u*-fields as

$$u_0(x) = v_0 + \frac{\mathrm{i}g}{\xi + \mathrm{i}A_0}, \tag{108}$$

$$u_{\pm 1}(x) = \mp \pi g \rho_0 - \frac{\mathrm{i}g}{\xi \pm \mathrm{i}A_1}, \tag{109}$$

where $u_{\pm 1}(x) = u_1(x \pm \mathrm{i}0)$. It also has a simple one-pole form in terms of the field (41)

$$\Phi = \sqrt{\rho_0} \frac{\xi - \mathrm{i}A_0}{\xi + \mathrm{i}A_1} \, \mathrm{e}^{\frac{\mathrm{i}}{g}(v_0 + \pi g \rho_0)\xi} \, \mathrm{e}^{-\frac{\mathrm{i}}{2g}(v_0 + \pi g \rho_0)^2 t}. \tag{110}$$

7.2.3. Analyticity condition. Now we can turn to the multi-phase solution and derive conditions sufficient in order for u_1 to be analytic in the upper half-plane in the complex *x*-variable (inside the unit disk). Analyticity in the lower half-plane follows from the Schwarz reflection condition (54). We will follow the approach of Dobrokhotov and Krichever [25] developed for the Benjamin–Ono equation.

Analyticity of u_1 means that τ_1 given by (79) has no zeros in the upper half-plane, or that the matrix

$$M_{jk} = \delta_{jk} + \frac{c_{1,j} \, \mathrm{e}^{\mathrm{i}\theta_j}}{p_j - q_k} \tag{111}$$

is non-degenerate. Following the approach of [25] we derived in appendix E a sufficient condition of non-degeneracy of the matrix M from (111). Let us now write conditions (E.10) and (E.9) with c_j defined by (E.1) and (92). We obtain (calculating f_j)

$$\frac{P}{p_j(q_j-P)}\prod_{k(k\neq j)}\frac{p_j-q_k}{p_j-p_k} \quad \text{same sign for all } j, \tag{112}$$

with P from (90).

The set of conditions

$$q_1 < p_1 < \cdots < q_m < p_m < 0 < P < q_{m+1} < p_{m+1} < \cdots < q_N < p_N \tag{113}$$

satisfies (112). Moreover, (113) yields to (100), which in its turn means that at least at some values of parameters (large time and soliton limit) no zeros of τ_1 are inside the unit disk. Since they also cannot be on the circle they do not cross it while moving in time and in the space of parameters.

Condition (113) suggests that a general solution is characterized by a integer number $N-2m$. This is chirality—the difference between the number of $N-m$ right and m—left moving modes

$$\frac{1}{2\pi g}\int(J_R - v_0 - \pi g\rho_0)\,dx = N-m, \tag{114}$$

$$\frac{1}{2\pi g}\int(J_L - v_0 + \pi g\rho_0)\,dx = m. \tag{115}$$

Equations (79)–(83), (92) and (113) summarize a general finite dimensional quasiperiodic solution. We emphasize here that this solution is not chiral and contains both right- and left-moving modes.

7.3. Multi-phase solution of the chiral non-linear equation

The (right) chiral case appears when τ_0 has no zeros outside the unit disk. It naturally happens when the number of, say, left-moving modes m in (113) vanishes $m = 0$. In this case, all $v_j - v_0 < 0$. In their turn, imaginary parts of zeros of τ_0 in the multi-soliton limit (as in (99))

$$-A_{0j} = -g\frac{v_j - v_0}{(v_j-v_0)^2 - K^2} > 0 \tag{116}$$

are positive. One can check that in this case (111) with $c_{1,j} \to c_{0,j}$ is non-degenerate for arbitrary values of parameters satisfying (113) with $m = 0$ (and similarly for $m = N$). Therefore, $\tau_0(x)$ is non-zero in one of the half-planes.

This is a chiral multi-phase solution of 2BO.

7.4. Multi-phase solution of the Benjamin–Ono equation

The known solutions of the Benjamin–Ono equation [21, 22] are obtained from the solutions of the chiral nonlinear equation by taking the limit $\rho_0 \to \infty$. In this case, $K \to -\infty$ and conditions (113) allow for a good limit only if $m = N$ (left sector) or $m = 0$ (right sector). Let us concentrate on, e.g., the left sector. We choose $v_0 = -K$ and obtain from (113) and (92) in the limit $K \to -\infty$

$$q_1 < p_1 < q_2 < \cdots < q_N < p_N < 0, \tag{117}$$

$$\left(\frac{c_{a,j}}{p_j - q_j}\right)^2 \prod_{k\neq j} \frac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)} = \left(\frac{p_j}{q_j}\right)^{1-2a} \tag{118}$$

with solution given by (79) and (81) and with (82) and (83) (one should put $K = 0$ in latter two). This is nothing but the multi-phase solution of the conventional Benjamin–Ono equation [19, 22].

7.5. Moving poles

The 2BO equation (14) looks very similar to the classical BO equation. One of the important tools in studying the classical BO equation is the so-called pole ansatz—solutions in the form of poles moving in a complex plane [21]. We have already seen that the pole ansatz (8) and (9) describes the dynamics of the original Calogero–Sutherland model with finite number of particles *N*.

In this section, we consider collective excitations of the Calogero–Sutherland model in the limit of infinitely many particles. These excitations are given by 'complex' pole solutions of the 2BO.

In the pole ansatz (8) and (9), the reality conditions were satisfied by requiring x_j to be real (or $w_j(t)$ moving on a unit circle). One could generalize the pole ansatz (8) and (9) to case where $w_j(t)$ are away from the unit circle and moving in a complex plane. Equations (6) and (7) describing the motion of poles preserve their form. However, $u_{-1}(w)$ outside of the unit circle is not related to $u_1(w)$ inside of the circle by analytic continuation but only by Schwarz reflection (10). The field $u_1(w)$ is analytic inside the unit circle and has moving poles outside of the unit circle (and vice versa for $u_{-1}(w)$). Of course, having obtained the solution of 2BO inside the unit circle does not mean automatically that the Schwarz reflected function (10) will solve 2BO in the exterior of the circle *with the same* u_0. The property (10) requires that (6) and (7) be satisfied not only by u_j and w_j but also by u_j and $1/\bar{w}_j$. This requirement will significantly constrain the positions of poles w_j and u_j in a complex plane. It turns out that this constraint allows for non-trivial solutions.

We emphasize here once again that while real axis poles x_j of u_1 in the pole ansatz represent the original CS particles, the complex poles x_j represent collective excitations of the CS liquid moving in the background of macroscopic number of particles.

Instead of looking for a moving pole solution in this section, we have taken a different route. We first construct the much more general solution of 2BO (14) with proper reality conditions and then obtain a moving pole (i.e., multi-soliton) solution as a limit of the multi-phase solution. One can see from (96) that for soliton solutions the zeros of tau-functions move in a complex plane. It is especially clear at large times when solitons are well separated (98).

8. Conclusion and discussion

In this paper, we have shown that the dynamics of the classical Calogero–Sutherland model in the limit of an infinite number of particles is equivalent to the bidirectional Benjamin–Ono equation (14). The bidirectional Benjamin–Ono equation (2BO) is an integrable classical integro-differential equation. Its integrability can be deduced from the fact that it is a Hamiltonian reduction of MKP1 as it is shown in this paper. As an alternative, one can use the equivalence of 2BO to INLS (40). The integrability of INLS was proven and the spectral transform was constructed for INLS in [26] (see also [20]). Therefore, one can use all techniques developed in the field of classical integrable equations for 2BO. It has multi-phase solutions (explicitly constructed in this paper), bi-Hamiltonian structure, an associated hierarchy of higher order equations, etc. 2BO is intrinsically simpler than many other classical integrable models. Its solitons have 'quantized' area independent of soliton's velocity. The collision of two solitons goes without any time delay, etc. This is a reflection of the fact that the underlying Calogero–Sutherland model is essentially a model of non-interacting particles in disguise. In particular, 2BO supports a phenomenon of dispersive shock waves. Some applications of this phenomenon to the quasi-classical description of quantum systems were considered in [16].

Most of the results of this paper can be generalized along two avenues: generalization to an elliptic case and generalization to a quantum model.

The Calogero–Sutherland model (trigonometric case) can be generalized to an elliptic case—an elliptic Calogero model where the interaction between particles is a Weierstrass $\wp(x|\omega_1,\omega_2)$ function with purely real and purely imaginary periods $\omega_1, \mathrm{i}\omega_2$, and to its hyperbolic degeneration (hyperbolic case) with inter-particle interaction given by $\sinh^{-2}(x/\omega_2)$ (see [3] for review).

In both cases most of the formulae remain unchanged if one substitutes the Hilbert transform f^H for a transform with respect to a strip $0 < \operatorname{Im} x < \omega$, where ω is an imaginary period:

$$f^H = \int \zeta(x-x')f(x')\,\mathrm{d}x' \qquad \text{or} \qquad \int \frac{1}{\omega_2}\coth\frac{1}{\omega_2}(x-x')f(x')\,\mathrm{d}x'. \tag{119}$$

In the first case the integration goes over a real period of the Weierstrass ζ-function.

The elliptic Calogero model allows one to study a crossover between liquids with long-range inter-particle interaction to liquids with short-range interaction. In the limit of a large imaginary period $\omega_2 \to \infty$, the $\wp$-function degenerates to $1/\sin(x/\omega_1)^2$ —the case of long-range inter-particle interaction. The opposite limit $\omega_2 \to 0$ gives rise to a short-range interaction: $\omega_2\wp(x) \to \delta(x)$.

In the latter case, the Hilbert transform (119) becomes a derivative $f^H \to \omega\partial_x f$ and the equations discussed in this paper become local. In particular, the Benjamin–Ono equation flows to the KdV equation, while the bidirectional BO equation flows to NLS—the nonlinear Schrödinger equation.

2BO in the limit of small amplitudes and in the chiral sector becomes the conventional Benjamin–Ono equation. In elliptic case (and in the hyperbolic one) the limit of small amplitudes in the chiral sector leads to a generalization of the Benjamin–Ono equation, known as the ILW (intermediate long wave) equation [8]. Contrary to the Benjamin–Ono equation and to 2BO, the latter and its bidirectional generalization 2ILW have elliptic solutions.

We intend to address the elliptic case in a separate publication.

Probably, even more interesting is a generalization of the results of this paper to the quantum case. It is well known that the classical CSM model (1) can be lifted to a quantum integrable Calogero–Sutherland model [1, 2, 28]. The latter model is defined by (1) with $g^2 \to \hbar^2\lambda(\lambda-1)$ and $p_i = -\mathrm{i}\hbar\partial_{x_i}$. The 2BO equation in the form (14) remains unchanged, except for the change of the coefficient $g \to \lambda - 1$ and for the change of Poisson brackets (47) by a commutator: $\{,\} \to \frac{\mathrm{i}}{\hbar}[,]$. The change $g \to \lambda - 1$ valid for equation (14) is not correct for all formulae. For example, the bilinear form of classical 2BO (49) is identical to its quantum version with just a change of notations $g \to \lambda$. For some details see [4]. The multi-soliton solution of 2BO presented here corresponds to exact quasiparticle excitations of the quantum Calogero–Sutherland model [7, 30]. A more detailed study of the relations between integrable structures of the classical 2BO and its quantum analog is necessary.

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Appendix A. Hilbert transforms

Given a function $f(x)$, $f(x) \to 0$ as $x \to \pm\infty$, the Hilbert transform is defined as

$$f^H(x) = \frac{1}{\pi} \fint_{-\infty}^{+\infty} \mathrm{d}y \frac{f(y)}{y-x}. \tag{A.1}$$

For periodic functions with period L we define the transform as

$$f^H(x) = \fint_0^L \frac{\mathrm{d}y}{L} f(y) \cot \frac{\pi}{L}(y-x). \tag{A.2}$$

The Hilbert transform of the constant function is zero. The Hilbert transform is inverse to itself $H^2 = -1$ or $(f^H)^H = -f$ and it commutes with derivative $(f^H)_x = (f_x)^H$. More generally, a function $f(x)$ defined on the closed (directed) contour C surrounded the origin of a complex plane can be decomposed into a sum $f = f^+ + f^-$, analytic functions $f^\pm$ inside (outside) of the contour, such that $f^+(0) = f^-(\infty)$. Then

$$f^H \equiv P \oint \frac{\mathrm{d}\zeta}{2\pi\zeta} f(\zeta) \frac{\zeta + w}{\zeta - w} = \mathrm{i}(f^+ - f^-). \tag{A.3}$$

Using (A.3) it is easy to derive the following properties:

$$f^H g + f g^H = (fg)^H - (f^H g^H)^H \tag{A.4}$$

and some integration formulae

$$\int \mathrm{d}x\, f^H = 0, \tag{A.5}$$

$$\int \mathrm{d}x\, f^H g = -\int \mathrm{d}x\, f g^H, \tag{A.6}$$

$$\int \mathrm{d}x\, f^H f = 0. \tag{A.7}$$

From (A.3) we have for functions analytic in one of the half-planes

$$(f^\pm)^H = \pm \mathrm{i} f^\pm \tag{A.8}$$

and as an immediate consequence

$$(\mathrm{e}^{\mathrm{i}kx})^H = \mathrm{i}\, \mathrm{e}^{\mathrm{i}kx} \operatorname{sgn} k, \tag{A.9}$$

$$\left(\frac{1}{x-a}\right)^H = \frac{\mathrm{i}}{x-a}, \qquad \text{for} \quad \operatorname{Im} a > 0. \tag{A.10}$$

Generally, in Fourier space the Hilbert transform is equivalent to a multiplication by $\mathrm{i} \operatorname{sgn} k$, i.e., for Fourier coefficients

$$(f^H)_k = \mathrm{i}(\operatorname{sgn} k) f_k. \tag{A.11}$$

It is easy to derive the following useful identities:

$$\int u^2\,\mathrm{d}x = \int (u^H)^2\,\mathrm{d}x, \tag{A.12}$$

$$\int u^3\,\mathrm{d}x = 3\int u(u^H)^2\,\mathrm{d}x, \tag{A.13}$$

$$\int u^4\,\mathrm{d}x = \int [(u^H)^4 + 6u^2(u^H)^2]\,\mathrm{d}x, \tag{A.14}$$

$$\int u^5\,\mathrm{d}x = 5\int [u(u^H)^4 - 2u^3(u^H)^2]\,\mathrm{d}x. \tag{A.15}$$

Appendix B. Conserved integrals of 2BO

In this section, we present the conserved integrals of 2BO written in different forms. The contour integrals below are taken along the contour C defined in figure 1:

$$I_1 = \oint \frac{\mathrm{d}x}{2\pi g} u = \int \mathrm{d}x\,\rho = \int \frac{\mathrm{d}x}{2\pi}\mathrm{i}\partial_x \log\frac{\tau_1}{\tau_{-1}}, \tag{B.1}$$

$$\begin{aligned} I_2 &= \oint \frac{\mathrm{d}x}{2\pi g}\frac{1}{2}u^2 = \oint \frac{\mathrm{d}x}{2\pi g} u_1 u_0 = \int \mathrm{d}x\,\rho v \\ &= \frac{g}{2}\int \frac{\mathrm{d}x}{2\pi}\left[\frac{D_x^2\tau_{-1}\cdot\tau_0}{\tau_{-1}\tau_0} - \frac{D_x^2\tau_1\cdot\tau_0}{\tau_1\tau_0}\right] = \mathrm{i}\partial_t\int\frac{\mathrm{d}x}{2\pi}\log\frac{\tau_1}{\tau_{-1}}, \end{aligned} \tag{B.2}$$

$$\begin{aligned} I_3 &= \oint \frac{\mathrm{d}x}{2\pi g}\left[\frac{1}{3}u^3 + \mathrm{i}\frac{g}{2}u\partial_x\tilde{u}\right] \\ &= \oint \frac{\mathrm{d}x}{2\pi g}\left[{u_1}^2 u_0 + u_1 {u_0}^2 + \mathrm{i}g u_1 \partial u_0\right] = \int \mathrm{d}x\left[\frac{\rho v^2}{2} + \rho\epsilon(\rho)\right]. \end{aligned} \tag{B.3}$$

Higher integrals of motion for 2BO can be constructed recurrently similarly to the Benjamin–Ono equation [27] or can be written using the integrals obtained for INLS [20, 26].

Appendix C. Geometrical interpretation of the chiral equation: contour dynamics

Equation (66) can be cast in the form of contour dynamics.

Let us interpret the unit disk as a uniformization of a simply connected domain embedded into the complex z-plane. In other words, $z(w)$ is a conformal map of the interior of the unit disk $|w| = 1$ to a bounded domain such that the length element of its boundary is proportional to the density

$$\mathrm{d}s \equiv |z'(x)|\,\mathrm{d}x = \pi\rho(x)\,\mathrm{d}x. \tag{C.1}$$

(Equivalently $1/\sqrt{\rho}$ is a harmonic measure of the contour.) Then equation (66) describes the evolution of the planar domain. We note that the curvature of the boundary $\kappa = -\mathrm{i}\partial_s \log z_s$ can be expressed in terms of the density $\rho(x)$ as

$$\kappa = -(\pi\rho)^{-1}(\partial_x(\log\sqrt{\rho})^H - 1). \tag{C.2}$$

Then (66) can be written as

$$\frac{\mathrm{d}s}{\mathrm{d}t} = g\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 (1-\kappa), \tag{C.3}$$

where $\mathrm{d}/\mathrm{d}t = \partial_t - g\pi\rho_0\partial_x$ and the time derivative is taken at fixed x. Equation (C.3) describes the evolution of a planar contour driven by its curvature.

Appendix D. Determinant identity

Here we present a derivation of a determinant identity (D.7) which was used in section 7.2.

Consider the Cauchy matrix

$$D_{ij} = \frac{r_i s_j}{p_i - q_j}, \qquad p_i \neq q_j, \quad i, j = 1, 2, \ldots, N. \tag{D.1}$$

Its determinant

$$\det(D) = \prod_i r_i \frac{\prod_{i<j}(p_i - p_j)(q_j - q_i)}{\prod_{i,j}(x_i - y_j)} \prod_j s_j = \left(\prod_i r_i s_i \tilde{r}_i \tilde{s}_i\right)^{-\frac{1}{2}}, \tag{D.2}$$

where $\tilde{r}_i$ and $\tilde{s}_i$ are defined by

$$\tilde{r}_i r_i = (p_i - q_i) \prod_{k(k\neq i)} \frac{p_i - q_k}{p_i - p_k}, \tag{D.3}$$

$$\tilde{s}_i s_i = -(q_i - p_i) \prod_{k(k\neq i)} \frac{q_i - p_k}{q_i - q_k}. \tag{D.4}$$

The inverse of D is also a Cauchy matrix (see, e.g., [29]) given by

$$(D^{-1})_{ij} = \frac{\tilde{r}_j \tilde{s}_i}{p_j - q_i}. \tag{D.5}$$

The obvious identity

$$\det(1 + D) = \det(D)\det(1 + (D^T)^{-1}) \tag{D.6}$$

being specialized for Cauchy matrix D (D.1) reads

$$\frac{\det\left(\delta_{ij} + \frac{r_i s_j}{p_i - q_j}\right)}{\left(\prod_i r_i s_i\right)^{1/2}} = \frac{\det\left(\delta_{ij} + \frac{\tilde{r}_i \tilde{s}_j}{p_i - q_j}\right)}{\left(\prod_i \tilde{r}_i \tilde{s}_i\right)^{1/2}}. \tag{D.7}$$

Appendix E. Non-degeneracy condition of matrix (111)

In this section, we will derive the non-degeneracy condition for the matrix (111) following [25].

Let us introduce

$$c_j = c_{1,j} \exp\left\{\frac{\mathrm{i}}{g}\left[(q_j - p_j)x - \frac{q_j^2 - p_j^2}{2}t\right]\right\}. \tag{E.1}$$

Our aim is to derive a sufficient condition on coefficients c_j for the matrix

$$M_{jk} = \delta_{jk} + \frac{c_j}{p_j - q_k} \tag{E.2}$$

to be non-degenerate for real p_j and q_j.

Let us assume that the matrix (E.2) is degenerate. It means that the following equation has a non-zero solution:

$$r_j + \sum_k \frac{c_j}{p_j - q_k} r_k = 0. \tag{E.3}$$

We introduce the meromorphic function

$$\psi(z) = \sum_k \frac{r_k}{z - q_k} \tag{E.4}$$

and rewrite the condition (E.3) as

$$r_j = \mathrm{res}_{z=q_j}\psi(z) = -c_j\psi(p_j). \tag{E.5}$$

Now, let us consider the function

$$f(z) = \psi(z)\overline{\psi(\bar{z})}\prod_j \frac{z - q_j}{z - p_j}. \tag{E.6}$$

Here explicitly

$$\overline{\psi(\bar{z})} = \sum_k \frac{\bar{r}_k}{z - q_k}. \tag{E.7}$$

The function $f(z)$ is meromorphic and its residue at infinity is zero (as $\psi(z) \sim z^{-1}$ as $z \to \infty$). On the other hand (p_j and q_j are real numbers)

$$\begin{aligned} \mathrm{res}_{z=p_j} f + \mathrm{res}_{z=q_j} f &= \psi(p_j)\overline{\psi(p_j)}(p_j - q_j)\prod_{k(k\neq j)} \frac{p_j - q_k}{p_j - p_k} \\ &+ r_j\bar{r}_j \frac{1}{q_j - p_j}\prod_{k(k\neq j)}\frac{q_j - q_k}{q_j - p_k} = \frac{|r_j|^2}{|c_j|^2} f_j, \end{aligned} \tag{E.8}$$

where we introduced the notation

$$f_j = (p_j - q_j)\prod_{k(k\neq j)}\frac{p_j - q_k}{p_j - p_k}\left[1 - \frac{|c_j|^2}{(p_j - q_j)^2}\prod_{k(k\neq j)}\frac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)}\right]. \tag{E.9}$$

If f_j has the same sign for all j, e.g., $f_j < 0$ we immediately obtain

$$\sum_j (\mathrm{res}_{z=p_j} f + \mathrm{res}_{z=q_j} f) < 0.$$

This is impossible as the residue of $f(z)$ at infinity is zero. This contradiction shows that the matrix (E.2) is non-degenerate under these conditions.

To summarize, the sufficient condition of non-degeneracy of (E.2) is

$$f_j < 0 \text{ for all } j \qquad \text{or} \qquad f_j > 0 \text{ for all } j, \tag{E.10}$$

where f_j are defined according to (E.9).

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